ON ISOMORPHISMS BETWEEN QUIVER VARIETIES OF TYPE A AND SLICES IN THE AFFINE GRASSMANNIAN

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The main references are [MV], [BF1] and also [Hen].

1. Examples of symplectic resolutions

There are three families of symplectic resolutions: Nakajima quiver varieties, Slodowy varieties and transversal slices in affine Grassmannians. In this talk, we will concentrate on two of these families – quiver varieties and slices in affine Grassmannian. The main result of this talk is isomorphisms between these families in type A. These isomorphisms can be thought of as a geometric incarnation of a Howe duality.

Remark 1.1. *It follows from the results of Maffei (*[Maf]*) that in type A Slodowy varieties are isomorphic to quiver varieties of type A so these three families of symplectic resolutions coincide in type A.*

1.1. **Nakajima quiver varieties.** We start with a quiver $Q = (I, \Omega)$ with vertices I and arrows Ω.

Passing to the Nakajima (framed) version of the quiver involves first doubling the arrows to $H = \Omega \sqcup \overline{\Omega}$ where $\Omega \xrightarrow{\sim} \overline{\Omega}$, $\omega \mapsto \overline{\omega}$, is the reversal of orientation. For an arrow $h \in H$ we denote by $h' \in I$ its initial vertex and by $h'' \in I$ its terminal vertex.

The data for framed quiver varieties are two I-graded vector spaces $V = \oplus_{i\in I} V_i$ and $D = \oplus_{i\in I} D_i$. Their dimension vectors $\underline{v}, \underline{d} \in \mathbb{N}^I$ define a vector space

$$M(\underline{v}, \underline{d}) \;=\; \bigoplus_{h\in H} \mathrm{Hom}(V_{h'}, V_{h''}) \oplus \bigoplus_{i\in I} \mathrm{Hom}(D_i, V_i) \oplus \bigoplus_{i\in I} \mathrm{Hom}(V_i, D_i).$$

We will consider an element in $M(\underline{v}, \underline{d})$ as a quadruple $(x, \overline{x}, p, q)$ with

$$\begin{aligned} x = (x_h)_{h\in\Omega} &\in \bigoplus_{h\in\Omega} \mathrm{Hom}(V_{h'}, V_{h''}), & \overline{x} = (x_h)_{h\in\overline{\Omega}} &\in \bigoplus_{h\in\overline{\Omega}} \mathrm{Hom}(V_{h'}, V_{h''}), \\ p = (p_i)_{i\in I} &\in \bigoplus_{i\in I} \mathrm{Hom}(D_i, V_i), & q = (q_i)_{i\in I} &\in \bigoplus_{i\in I} \mathrm{Hom}(V_i, D_i). \end{aligned} \tag{1.1}$$

The group $G(V) := \prod_{i\in I} \mathrm{GL}(V_i)$ acts naturally on $M(\underline{v}, \underline{d})$. This action is Hamiltonian. The corresponding moment map

$$\mu \colon M(v, d) \to \mathfrak{g}(V) \overset{\text{def}}{=} Lie[G(V)] \cong \oplus gl(V_i),$$

is given by

$$(x, \overline{x}, p, q) \mapsto [x, \overline{x}] + pq.$$

The affine invariant theory quotient of $\mu^{-1}(0)$ by $G(V)$ is denoted

$$\mathfrak{M}(\underline{v}, \underline{d}) := \mu^{-1}(0) /\!\!/ G(V) = \mathrm{Spec}(\mathbb{C}[\mu^{-1}(0)]^{G(V)}).$$

This is an *affine* Nakajima quiver variety corresponding to $(I, \Omega, \underline{v}, \underline{d})$. This is a Poisson singular (in general) affine variety.

One can consider the character $\det\colon G(V) \to \mathbb{C}^\times$ and construct the corresponding GIT-quotient

$$\mathfrak{M}(\underline{v}, \underline{d}) := \mu^{-1}(0)/\!\!/_{\det} G(V) = \mu^{-1}(0)^{st}/G(V),$$

where $\mu^{-1}(0)^{st} \subset \mu^{-1}(0)$ is the subset of det-stable points that can be described as follows.

A quadruple $(x, \overline{x}, p, q)$ is det-stable iff for any I-graded subspace $V' \subset V$ such that

$$x(V') \subset V',\, x(V) \subset V',\, \operatorname{im} p \subset V$$

we have $V' = V$.

Variety $\mathfrak{M}(\underline{v}, \underline{d})$ is a symplectic resolution of singularities (in particularly it is smooth and symplectic).

Example 1.2. *For a quiver* A_{N-1} *and* $\underline{v} = (N-1, N-2, \ldots, 1)$, $\underline{d} = (N, 0, 0 \ldots, 0)$ *symplectic variety* $\mathfrak{M}(\underline{v}, \underline{d})$ *is nothing else but* $T^*\mathcal{F}_N$*:*

$$\mathfrak{M}(\underline{v}, \underline{d}) \xrightarrow{\sim} T^*\mathcal{F}_N,\, (x, \overline{x}, p, q) \mapsto (q_1p_1, \{0\} \subset \ker p \subset \ker x_1p_1 \subset \ldots \subset \ker x_{n-1} \ldots x_1p_1),$$

here $x_i\colon V_i \to V_{i+1}$, $p_1\colon D_1 \to V_1$, $q_1\colon V_1 \to D_1$. *The variety* $\mathfrak{M}_0(\underline{v}, \underline{d})$ *is isomorphic to* $\mathcal{N} \subset \mathfrak{gl}_N$*:*

$$\mathfrak{M}(\underline{v}, \underline{d}) \xrightarrow{\sim} \mathcal{N},\, (x, \overline{x}, p, q) \mapsto q_1p_1.$$

Note that in general we have the natural projective morphism

$$\mathbf{p}\colon \mathfrak{M}(\underline{v}, \underline{d}) \to \mathfrak{M}_0(\underline{v}, \underline{d}).$$

This morphism is an isomorphism over some open subvariety $\mathfrak{M}_0^{reg}(\underline{v}, \underline{d}) \subset \mathfrak{M}_0(\underline{v}, \underline{d})$ that can be described as follows.

We say that a quadruple $(x, \overline{x}, p, q)$ is costable (stable for $\det^{-1}$) if for any I-graded subspace $V' \subset V$ such that

$$x(V') \subset V',\, \overline{x}(V') \subset V',\, V' \subset \ker q$$

we have $V' = 0$. The open subvariety $\mathfrak{M}_0^{reg}(\underline{v}, \underline{d}) \subset \mathfrak{M}_0(\underline{v}, \underline{d})$ by the definition consists of quadruples that are both stable and costable.

Proposition 1.3. *Morphism* $\mathbf{p}$ *induces an isomorphism* $\mathbf{p}^{-1}(\mathfrak{M}_0^{reg}(\underline{v}, \underline{d})) \xrightarrow{\sim} \mathfrak{M}_0^{reg}(\underline{v}, \underline{d})$.

Remark 1.4. *Note that the variety* $\mathfrak{M}_0^{reg}(\underline{v}, \underline{d})$ *may be empty. For a quiver* A_1, $\underline{v} = v \in \mathbb{N}$, $\underline{d} = d \in \mathbb{N}$ *our data is*

$$p\colon \mathbb{C}^d \to \mathbb{C}^v,\, q\colon \mathbb{C}^v \to \mathbb{C}^d,\, pq = 0.$$

Then the pair (p, q) *is stable iff* p *is surjective and is costable iff* q *is injective. Note also that* $\operatorname{im} q \subset \ker p$. *So for regular (stable and costable) point we have*

$$q\colon \mathbb{C}^v \hookrightarrow \ker p \hookrightarrow \mathbb{C}^d \xrightarrow{p} \mathbb{C}^v \to 0$$

but this is possible only if $d \geqslant 2v$.

Nakajima gave a general combinatorial criteria for $\mathfrak{M}^{reg}(\underline{v}, \underline{d})$ to be nonempty (generalising the condition $d \geqslant v$ from remark 1.4). Let us formulate this criteria for quivers Q of finite type (ADE quiver). Let $\mathfrak{g}$ be the Lie algebra corresponding to Q.

Proposition 1.5. *Variety* $\mathfrak{M}^{reg}(\underline{v}, \underline{d})$ *is nonempty iff* $\underline{d} - C\underline{v} \in \mathbb{Z}^I_{\geqslant 0}$ *and* $\Lambda_{\underline{d}} - \alpha_{\underline{v}}$ *is a weight of an integrable highest weight module* $V(\mathfrak{g})^{\Lambda_{\underline{d}}}$ *over* $\mathfrak{g}$ *with highest weight* $\Lambda_{\underline{d}}$, *here* $\Lambda_{\underline{d}} := \sum_{i \in I} d_i\omega_i$, $\alpha_{\underline{v}} := \sum_{i \in I} v_i\alpha_i$, ω_i, α_i *are fundamental and simple roots of* $\mathfrak{g}$ *and* C *is the Cartan matrix of* $\mathfrak{g}$.

Remark 1.6. *Note that if $\Lambda_{\underline{d}} - \alpha_{\underline{v}}$ is dominant then it is automatically a weight of $V(\mathfrak{g})^{\Lambda_{\underline{d}}}$.*

Let us now recall the representation-theoretic application of quiver varieties. Set $\mathcal{L}(\underline{v}, \underline{d}) := \mathbf{p}^{-1}(0) \subset \mathfrak{M}(\underline{v}, \underline{d})$.

Proposition 1.7. *Fix $\underline{d} \in \mathbb{Z}^I_{\geqslant 0}$. Vector space $\oplus_{\underline{v} \in \mathbb{Z}^I_{\geqslant 0}} H^{BM}_{\mathrm{top}}(\mathcal{L}(\underline{v}, \underline{d}))$ has a structure of integrable simple $\mathfrak{g}$-module with highest weight $\Lambda_{\underline{d}}$. The component $H^{BM}_{\mathrm{top}}(\mathcal{L}(\underline{v}, \underline{d}))$ is the weight space of weight $\Lambda_{\underline{d}} - \alpha_{\underline{v}}$.*

1.2. **Slodowy varieties.** We will not talk about Slodowy varieties later. Let us just mention that these varieties depend on a pair of a nilpotent element in a Lie algebra $\mathfrak{g}$ of a reductive group G and a parabolic subgroup $P \subset G$. For $e = 0$ the corresponding Slodowy variety is $T^*(G/P)$. In general, this is a closed subvariety in $T^*(G/P)$.

Let us also mention that in type A the corresponding simplectic resolution $T^*(G/P) \to \mathrm{Spec}(\mathbb{C}[T^*(G/P)])$ coincides with the resolution $T^*(G/P) \to \overline{O}$, where $O \subset \mathcal{N}$ is a nilpotent orbit in the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ of $\mathfrak{g}$ corresponding to P.

In the simplest case $P = B$ i.e. when P is a Borel subgroup our Slodowy variety is nothing else but $T^*(G/B)$ which resolves nilpotent cone $\mathcal{N}$.

Recall that by Example 1.2 in type A we have an isomorphism of symplectic resolutions $T^*(\mathcal{F}_N) \simeq \mathfrak{M}(\underline{v}, \underline{d})$, where $\underline{v} = (N-1, \ldots, 1)$, $\underline{d} = (N, 0, \ldots, 0)$ and $\mathcal{F}_n$ is the flag variety for GL_N.

More generally in type A we have isomorphisms

$$\mathfrak{M}(\underline{v}, \underline{d}) \xrightarrow{\sim} T^*(\mathcal{F}^{\underline{a}}_N),\ (x, \overline{x}, p, q) \mapsto (q_1 p_1, \{0\} \subset \ker p \subset \ker x_1 p_1 \subset \ldots \subset \ker x_{n-1} \ldots x_1 p_1),$$

where $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 0}$, $a_1 + \ldots + a_n = N$, $\mathcal{F}^{\underline{a}}_N$ is the variety of partial flags $\{0\} = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = D$ of subspaces of D such that $\dim F_i - \dim F_{i-1} = a_i$. We have $\underline{d} = (N, 0, \ldots, 0)$ and $\underline{v} = (N - a_1, N - a_1 - a_2, \ldots, N - a_1 - \ldots - a_{n-1})$.

These are simple cases of Maffei's isomorphisms between Slodowy varieties of type A and quiver varieties of type A. See [Maf].

1.3. **Affine Grassmannian and transversal slices.**

1.3.1. *Affine Grassmannian.* Let Gr_G be the moduli space of G-bundles $\mathcal{P}$ over $\mathbb{P}^1$ with a trivialization σ outside 0.

The space Gr_G can be defined as follows: set $\mathcal{K} := \mathbb{C}((z))$, $\mathcal{O} := \mathbb{C}[[z]]$, then Gr_G is the quotient $G(\mathcal{K})/G(\mathcal{O})$. Any cocharacter $\mu \in \Lambda$ gives rise to an element of Gr_G to be denoted by z^μ. The group $G(\mathcal{O})$ acts on Gr_G via left multiplication. For $\mu \in \Lambda^+$, denote by Gr^μ_G the $G(\mathcal{O})$-orbit of z^μ. We have the following decompositions:

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}^\lambda_G,\ \overline{\mathrm{Gr}}^\mu_G = \bigsqcup_{\lambda \leqslant \lambda} \mathrm{Gr}^\mu_G. \tag{1.2}$$

It is known that for any $\mu \in \Lambda^+$ $\overline{\mathrm{Gr}}^\mu$ is a projective algebraic variety of dimension $\langle 2\rho^\vee, \mu \rangle$, here $2\rho^\vee$ is the sum of positive roots. It follows that $\mathrm{Gr}_G = \varinjlim \overline{\mathrm{Gr}}^\mu$ is an ind-projective scheme.

We have an action $\mathbb{C}^\times \curvearrowright \mathrm{Gr}_G$ via loop rotation:

$$t \cdot g := (z \mapsto g(tz)).$$

The fixed points of $\mathbb{C}^\times \curvearrowright \mathrm{Gr}_G$ are $\bigsqcup_{\mu \in \Lambda^+} G z^\mu$.

It is easy to see that

$$\mathrm{Gr}^\mu_G = \{x \in \mathrm{Gr}_G \mid \lim_{t \to 0} t \cdot x \in G z^\mu\}$$

so in other words Gr^μ_G is *attractor* to Gz^μ w.r.t. the loop rotation.

1.3.2. *Transversal slices.* Variety Gr_G has another decomposition ("opposite" to (1.2)) corresponding to $G[z^{-1}]$-orbits.

For any $\lambda \in \Lambda$ set $\mathrm{Gr}_{G,\lambda} := G[z^{-1}]z^\lambda$. Then it follows from the Grothendieck theorem about classification of G-bundles on $\mathbb{P}^1$ that we have

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}_{G,\lambda}\,.$$

Directly by the definitions

$$\mathrm{Gr}_{G,\lambda} = \{x \in \mathrm{Gr}_G \mid \lim_{t\to\infty} t \cdot x \in Gz^\lambda\}.$$

Note that $\mathrm{Gr}_{G,\lambda} \cap \mathrm{Gr}_G^\lambda = Gz^\lambda$.

Let us denote by $G[z^{-1}]_1 \subset G[z^{-1}]$ the kernel of the natural evaluation at infinity homomorphism $G[z^{-1}] \to G$.

We set $\mathcal{W}_\lambda := G[z^{-1}]_1 z^\lambda$. By the definitions

$$\mathcal{W}_\lambda = \{x \in \mathrm{Gr}_G \mid \lim_{t\to\infty} t \cdot x = z^\lambda\},\ \mathcal{W}_\lambda \cap \mathrm{Gr}_G^\lambda = \{z^\lambda\}.$$

We can now finally define transversal slices as follows. For $\lambda \leqslant \mu$ (otherwise $\overline{\mathcal{W}}^\mu_\lambda$ will be empty) we set

$$\mathcal{W}^\mu_\lambda := \mathrm{Gr}^\mu_G \cap \mathcal{W}_\lambda,\ \overline{\mathcal{W}}^\mu_\lambda := \overline{\mathrm{Gr}}^\mu_G \cap \mathcal{W}_\lambda$$

It follows from the definitions that

$$\overline{\mathcal{W}}^\mu_\lambda = \{x \in \mathrm{Gr}_G \mid \lim_{t\to\infty} t \cdot x = z^\lambda,\ \lim_{t\to 0} t \cdot x \in \overline{\mathrm{Gr}}^\mu_G\}.$$

Variety $\overline{\mathcal{W}}^\mu_\lambda$ is an affine variety of dimension $\langle 2\rho^\vee, \mu - \lambda\rangle$ equipped with a contracting action of $\mathbb{C}^\times$, $\mathcal{W}^\mu_\lambda \subset \overline{\mathcal{W}}^\mu_\lambda$ is an open smooth subvariety.

Remark 1.8. *Variety $\overline{\mathcal{W}}^\mu_\lambda$ is a transversal slice to Gr^λ_G inside $\overline{\mathrm{Gr}}^\mu_G$ at the point z^μ in the following sence: there exists an open subset $U \subset \mathrm{Gr}^\lambda_G$ and an open embedding $U \times \overline{\mathcal{W}}^\mu_\lambda \hookrightarrow \overline{\mathrm{Gr}}^\mu_G$ such that the following diagram is commutative:*

$$\begin{array}{ccc} U \times \{z^\lambda\} & \longrightarrow & \mathrm{Gr}^\lambda_G \times \{z^\lambda\} \\ \downarrow & & \downarrow \\ U \times \overline{\mathcal{W}}^\mu_\lambda & \longrightarrow & \overline{\mathrm{Gr}}^\lambda_G. \end{array}$$

Using this, one can identify stalks of IC-sheaves of $\overline{\mathrm{Gr}}^\mu_G$ and $\overline{\mathcal{W}}^\mu_\lambda$ at the point z^λ.

Remark 1.9. *Variety $\overline{\mathcal{W}}^\mu_\lambda$ can be equipped with a Poisson structure comming from a Poisson structure on Gr_G.*

Remark 1.10. *Let us now relate Gr_G with representation theory. Let $G^\vee$ be the Langlands dual group to G. We denote by $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr_G. There is an equivalence of tensor categories $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \simeq \mathrm{Rep}_{fd}(G^\vee)$ with fiber functor for $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ given by $\mathcal{P} \mapsto H^*(\mathrm{Gr}_G, \mathcal{P})$. Tensor product on $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is defined using so-called convolution diagram.*

2. Main theorem

For now on our quiver variety is always of type A ($Q = A_{n-1}$). We identify vertices with integers $1, 2, \ldots, n-1$. Our quadruple $(x, \overline{x}, p, q) \in M(\underline{v}, \underline{d})$ consists of

$$x_i \colon V_i \to V_{i+1},\, \overline{x}_i \colon V_{i+1} \to V_i,$$

$$p_i \colon D_i \to V_i,\, q_i \colon V_i \to D_i.$$

We have an affine quiver variety $\mathfrak{M}_0(\underline{v}, \underline{d})$, we also set $D := \oplus D_i$, $V = \oplus_i V_i$ and pick a maximal torus $T \subset \mathrm{GL}(V)$.

Let λ be a dominant cocharacter of T that acts with eigenvalue t^i on D_i. Let μ be a cocharacter of T that acts with eigenvalue t^i on the subspace of D of dimension $v_{i+1} + v_{i-1} + d_i - 2v_i$. In particular we assume that $v_{i+1} + v_{i-1} + d_i - 2v_i \geqslant 0$ for any i. One can see that this is nothing else but the condition from Proposition 1.5 implying that $\mathfrak{M}_0^{reg}(\underline{v}, \underline{d}) \neq \varnothing$.

Remark 2.1. Recall that in Section 1.1 we associated to $\underline{v}, \underline{d}$ the following characters of T: $\Lambda_{\underline{d}} = \sum_i d_i \omega_i$, $\Lambda_{\underline{d}} - \lambda_{\underline{v}} = \Lambda_{\underline{d}} - \sum_i v_i \alpha_i$. Let ϵ_i be the standard basis of $\mathrm{Lie}\, \mathfrak{t}$. Then

$$\Lambda_{\underline{d}} = \sum_i x_i \epsilon_i,\, \Lambda_{\underline{d}} - \lambda_{\underline{v}} = \sum_i a_i \epsilon_i,$$

where

$$x_i = d_i + d_{i+1} + \ldots + d_{n-1},\, a_i = x_i + v_{i-1} - v_i.$$

We see that $a_i - a_{i+1} = v_{i+1} + v_{i-1} + d_i - 2v_i$ so our condition $v_{i+1} + v_{i-1} + d_i - 2v_i \geqslant 0$ say precisely that $\Lambda_{\underline{d}} - \sum_i v_i \alpha_i$ is dominant. It now follows from the definitions that

$$\Lambda_{\underline{d}} = \lambda^t,\, \Lambda_{\underline{d}} - \lambda_{\underline{v}} = \mu^t$$

considered as Young diagrams.

Theorem 2.2. *We have an isomorphism*

$$\Phi \colon \mathfrak{M}_0(\underline{v}, \underline{d}) \xrightarrow{\sim} \overline{\mathcal{W}}^{-w_0(\mu)}_{-w_0(\lambda)}$$

given by

$$(x, \overline{x}, p, q) \mapsto z^{-w_0(\lambda)}(1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-n} q \overline{x}^n x^l p) = z^{-w_0(\lambda)}(1 + q(\overline{x} - z)^{-1}(x-1)^{-1} p).$$

Remark 2.3. *We also have an isomorphism*

$$\mathfrak{M}_0(\underline{v}, \underline{d}) \xrightarrow{\sim} \overline{\mathcal{W}}^{\mu}_{\lambda}$$

given by

$$(x, \overline{x}, p, q) \mapsto z^{\lambda}(\mathrm{Id} + z^{-1} \sum_{n,l} (-z)^{-n} q x^n \overline{x}^l p) = z^{\lambda}(1 + q(x+z)^{-1}(\overline{x} - 1)^{-1} p).$$

The existence of two isomorphisms corresponds to the isomorphism $\mathfrak{M}(\underline{v}, \underline{d}) \xrightarrow{\sim} \mathfrak{M}(\underline{v}^{\dagger}, \underline{d}^{\dagger})$ $(x, \overline{x}, p, q) \mapsto (\overline{x}, -x, p, q)$, $v_i^{\dagger} = v_{-i}$, $d_i^{\dagger} = d_{-i}$.

Example 2.4. *Let us consider the simplest case:* $\underline{v} = (N-1, N-2, \ldots, 1)$, $\underline{d} = (N, 0, \ldots, 0)$. *Then* $\mathfrak{M}_0(\underline{v}, \underline{d}) \simeq \mathcal{N}$. *We also have* $\mu = (N, 0, 0, \ldots, 0)$, $\lambda = (1, 1, \ldots, 1)$ *so* $-w_0(\mu) = (0, 0, \ldots, -N)$, $-w_0(\lambda) = (-1, \ldots, -1)$. *Note that the element* $z^{-w_0(\lambda)}$ *is central so multiplication by* $z^{w_0(\lambda)} = z^{\lambda}$ *identifies* $\overline{\mathcal{W}}^{-w_0(\mu)}_{-w_0(\lambda)}$ *with* $\overline{\mathcal{W}}_0^{-w_0(\mu - \lambda)}$.

The corresponding embedding $\mathfrak{M}_0(\underline{v},\underline{d})\hookrightarrow \mathrm{Gr}_G$ *is given by*

$$(x,\overline{x},p,q)\mapsto 1+z^{-1}\sum_{l\geqslant 0} z^{-l}q_1\overline{x}^l x^l p_1.$$

It is easy to see (by induction on l *using that* $\mu(x,\overline{x},p,q)=0$*) that*

$$q_1\overline{x}^l x^l p_1 = q_1\overline{x}x\overline{x}^{l-1}x^{l-1}p_1 = q_1p_1q_1\overline{x}^{l-1}x^{l-1}p_1 = \ldots = (q_1p_1)^l.$$

So we obtain the following embedding

$$\mathcal{N}\hookrightarrow \mathrm{Gr}_G,\ x\mapsto 1+z^{-1}x+z^{-2}x^2+\ldots$$

Note that we have another embedding $\mathcal{N}\hookrightarrow\mathrm{Gr}_G$ *given by*

$$x\mapsto 1+z^{-1}x.$$

It exactly corresponds to our second isomorphism (see remark 2.3).

Let us mention that Mircović and Vybornov have constructed isomorphism

$$\mathfrak{M}(\underline{v},\underline{d})\xrightarrow{\sim}\widetilde{\mathcal{W}}^{-w_0(\mu)}_{-w_0(\lambda)}$$

of the corresponding symplectic resolutions. Here $\widetilde{\mathcal{W}}^{-w_0(\mu)}_{-w_0(\lambda)}$ is a symplectic resolution of $\overline{\mathcal{W}}^{-w_0(\mu)}_{-w_0(\lambda)}$ that can be constructed using convolution diagram for Grassmannians.

Remark 2.5. More detailed there exists a semi-small resolution $\pi\colon \widetilde{\mathrm{Gr}}_G^{-w_0\mu}\to\overline{\mathrm{Gr}}_G^{-w_0\mu}$ which is used to define tensor structure (convolution) on $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$. Then $\widetilde{\mathcal{W}}^{-w_0(\mu)}_{-w_0(\lambda)}=\pi^{-1}(\overline{\mathcal{W}}^{-w_0\mu}_{-w_0\lambda})$.

3. Idea of the proof

So our goal is to construct an isomorphism

$$\Phi\colon\mathfrak{M}_0(\underline{v},\underline{d})\xrightarrow{\sim}\overline{\mathcal{W}}^{-w_0\mu}_{-w_0\lambda}.$$

Let us for simplicity describe it on the open parts:

$$\Phi\colon\mathfrak{M}_0^{reg}(\underline{v},\underline{d})\xrightarrow{\sim}\mathcal{W}^{-w_0\mu}_{-w_0\lambda}.$$

Recall that $\mathcal{W}^{-w_0\mu}_{-w_0\lambda}$ is a moduli space of bundles together with a trivialization on $\mathbb{P}^1$ such that the limit of a loop rotation action when $t\to\infty$ is $z^{-w_0\lambda}$ and the limit when $t\to 0$ lies in $\overline{\mathrm{Gr}}_G^{-w_0\mu}$.

Note that these conditions can be said as follows: every point $x\in\mathcal{W}^{-w_0\mu}_{-w_0\lambda}$ defines us a unique morphism $u\colon\mathbb{P}^1\to\mathrm{Gr}_G$ such that u is $\mathbb{C}^\times$-equivariant, $u(1)=x$, $u(\infty)=z^{-w_0\lambda}$, $u(0)\in\overline{\mathrm{Gr}}_G^{-w_0\mu}$ and any such morphism defines us a point of $\mathcal{W}^{-w_0\mu}_{-w_0\lambda}$.

Note now that one can think about morphisms $u\colon\mathbb{P}^1\to\mathrm{Gr}_G$ as about vector bundles on $\mathbb{P}^1\times\mathbb{P}^1$ with trivializations on $\mathbb{P}^1\times(\mathbb{P}^1\setminus 0)$.

So we see that $\mathcal{W}^{-w_0\mu}_{-w_0\lambda}$ are certain (equivariant) vector bundles on $\mathbb{P}^1\times\mathbb{P}^1$ with trivialization.

It would be nice to find a simmilar description of $\mathfrak{M}^{reg}(\underline{v},\underline{d})$.

To do so let us recall the Gieseker variety and ADHM description of it.

3.1. **Gieseker variety and ADHM.** Let $\mathrm{Bun}^v_{GL_d}(\mathbb{A}^2)$ denote the moduli space of principal GL_v-bundles on $\mathbb{P}^1\times\mathbb{P}^1$ of second Chern class v with a trivialization at $\mathbb{P}^1\times\infty\cup\infty\times\mathbb{P}^1$.

Let V be a vector space of dimension v and D be a vector space of dimension d. We set $\mathfrak{M}_0^{reg}(V,D)=\{(x,\bar{x},p,q)\in\mu^{-1}(0)\mid \text{stable and costable }\}/GL_v$, where $(x,\bar{x},p,q)$ are Jordan quiver quadruples:

$$x,\overline{x}\colon V\to V,\ p\colon D\to V,\ q\colon V\to D.$$

The ADHM isomorphism identifies $\mathrm{Bun}^v_{GL_d}(\mathbb{A}^2)$ with $\mathfrak{M}_0^{reg}(V,D)$.

The vector bundle $E_{(x,\bar{x},p,q)}$ corresponding to a quadruple $(x,\bar{x},p,q)$ can be obtained as the middle cohomology of the following monad:

$$V\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)\xrightarrow{\ a\ }\begin{matrix}V\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1)\\ \bigoplus\\ V\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0)\\ \bigoplus\\ D\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\end{matrix}\xrightarrow{\ b\ }V\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$$

where

$$a=(kx-y,h\overline{x}-z,khq),\ b=(-(h\overline{x}-z),kx-y,p),$$

$([y:k],[z:h])$ are coordinates on $\mathbb{P}^1\times\mathbb{P}^1$ and $(\infty,\infty)=([1:0],[1:0])$.

3.2. **Variety $\mathfrak{M}^{reg}(\underline{v},\underline{d})$ as fixed points of $\mathfrak{M}^{reg}(V,D)$.** As we see from the ADHM description to relate $\mathfrak{M}^{reg}(\underline{v},\underline{d})$ with vector bundles on $\mathbb{P}^1\times\mathbb{P}^1$ it remains to understand the relation between the varieties $\mathfrak{M}^{reg}(\underline{v},\underline{d})$ and $\mathfrak{M}^{reg}(V,D)$.

Let $T\subset\mathrm{GL}(D)$ be a maximal torus. Note that we have a symplectic action $T\curvearrowright\mathfrak{M}^{reg}(V,D)$ given by

$$(x,\overline{x},p,q)\mapsto(x,\overline{x},p\circ u^{-1},u\circ q),\ u\in T.$$

We also have an action $\mathbb{C}^\times\curvearrowright\mathfrak{M}^{reg}(V,D)$ given by

$$(x,\overline{x},p,q)\mapsto(t^{-1}x,t\overline{x},p,q),\ t\in\mathbb{C}^\times.$$

These actions commute so we obtain an action

$$\mathbb{C}^\times\times T\curvearrowright\mathfrak{M}^{reg}(V,D).$$

Recall now that we have a co-character $-w_0\lambda\colon\mathbb{C}^\times\to T$. Consider a cocharacter

$$\rho_\lambda\colon\mathbb{C}^\times\to\mathbb{C}^\times\times T,\ t\mapsto(t,-w_0\lambda(t)).$$

Proposition 3.1. *We have an isomorphism*

$$\Theta\colon\bigsqcup_{\underline{v},\sum_i v_i=v}\mathfrak{M}_0^{reg}(\underline{v},\underline{d})\xrightarrow{\sim}(\mathfrak{M}_0^{reg}(V,D))^{\rho_\lambda(\mathbb{C}^\times)}$$

given by

$$\Theta(x_i,\overline{x}_i,p_i,q_i)\mapsto(\oplus_i x_i,\oplus_i\overline{x}_i,\oplus_i p_i,\oplus_i q_i).$$

Proof. We describe the inverse map. Let $(x,\bar{x},p,q)$ be a fixed point under the $\mathbb{C}^*$-action on $\mathfrak{M}_0^{reg}(V,D)$ corresponding to $-w_0(\lambda)$. Then for every $t\in\mathbb{C}^*$ there exists $\rho_V(t)\in GL(V)$ such that

$$(t^{-1}x, t\bar{x}, p(-w_0(\lambda)(t))^{-1}, -w_0(\lambda)(t)q) = (\rho_V(t)x\rho_V(t)^{-1}, \rho_V(t)\bar{x}\rho_V(t)^{-1}, \rho_V(t)p, q\rho_V(t)^{-1}). \tag{3.1}$$

Note that $\rho_V(t)$ is uniquely determined by t because of the freeness of $GL(V)$-action on stable and costable quadruples. In particular ρ_V defines a cocharacter of $GL(V)$. We decompose V into a direct sum $\oplus V_i$ (where V_i is the t^{-i}-eigenspace of ρ_V) and similarly decompose D into a direct sum $\oplus D_i$ with respect to $-w_0(\lambda)$. It is easy to see that the condition (3.1) implies that $\forall i\subset\mathbb{Z}$, $x(V_i)\subset V_{i+1}$, $\bar{x}(V_i)\subset V_{i-1}$, $p(D_i)\subset V_i$, $q(V_i)\subset D_i$. So $(x,\bar{x},p,q)$ defines a point in a quiver variety of type A with vertices numbered by integers such that $\sum\limits_{i=-\infty}^{+\infty} v_i = v$, and the framing is d. The inverse map is constructed.

□

4. More technical details

So the construction of the morphism $\Phi\colon \mathfrak{M}^{reg}(\underline{v},\underline{d}) \to \mathcal{W}^{-w_0(\mu)}_{-w_0(\lambda)}$ goes as follows. We start from a quadruple $(x_i,\overline{x}_i,p_i,q_i)\in\mathfrak{M}^{reg}(\underline{v},\underline{d})$ and associate to it (via $ADHM$) a $\mathbb{C}^\times$-equivariant vector bundle $E=E_{(x,\overline{x},p,q)}$ together with a trivialization at the cross $\mathbb{P}^1\times\infty\cup\infty\times\mathbb{P}^1$. Then $\mathbb{C}^\times$-equivariance of E allows us to uniquely extend the trivialization of E on $\mathbb{P}^1\times\infty$ to the trivialization of E on $\mathbb{P}^1\times(\mathbb{P}^1\setminus\{0\})$, hence, $E|_{1\times\mathbb{P}^1}$ defines a point of the Affine Grassmannian Gr_G to be denoted $\eta(E)$. Now

$$\Phi((x_i,\overline{x}_i,p_i,q_i)) := z^{-w_0\lambda}E|_{1\times\mathbb{P}^1}.$$

We see that if we want to compute Φ explicitly the only thing that we have to do is to understand how one can construct this extension of our trivialization of E on $\mathbb{P}^1\times\infty$ to the trivialization of E on $\mathbb{P}^1\times(\mathbb{P}^1\setminus\{0\})$ explicitly.

Monad description helps here: recall that E is the middle cohomology of the following monad:

$$V\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1)$$

$$\bigoplus$$

$$V\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1) \xrightarrow{\ a\ } V\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0) \xrightarrow{\ b\ } V\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$$

$$\bigoplus$$

$$D\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$$

where

$$a = (kx - y, h\overline{x} - z, khq),\ b = (-(h\overline{x} - z), kx - y, p),$$

$([y:k],[z:h])$ are coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ and $(\infty,\infty) = ([1:0],[1:0])$.

We want to describe the trivialization of E restricted to $\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)$. For this it suffices to construct a map $D \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)} \to \mathrm{Ker(b)}\,|_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}$ transversal to $\mathrm{Im(a)}\,|_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}$. It is easy to see that the map:

$$D \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)} \xrightarrow{\tau_1} V \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}(0,-1) \oplus V \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}(-1,-1) \oplus D \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)},$$

$$\tau_1 = ((h\overline{x} - z)^{-1} p, 0, \mathrm{Id})$$

satisfies the requirement.

Note that τ_1 is well defined because $h\bar{x}$ is nilpotent ($\bar{x} = \oplus \bar{x}_i$, and $\bar{x}_i$ sends V_i to V_{i-1}, so that $\oplus \bar{x}_i$ acts nilpotently on $\oplus V_i$), hence $h\overline{x} - z$ is invertible when restricted to $\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)$ (since $z \neq 0$ on $\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)$ and $h\bar{x}$ is nilpotent). We should mention that transvesality of τ_1 to the image of a follows from the fact that $h\overline{x} - z$ invertible, hence, nonzero for $z \neq 0$.

For the same reasons the map:

$$D \otimes \mathcal{O}_{(\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1} \xrightarrow{\tau_2} V \otimes \mathcal{O}_{(\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1}(0,-1) \oplus V \otimes \mathcal{O}_{(\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1}(-1,-1) \oplus D \otimes \mathcal{O}_{(\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1},$$

$$\tau_2 = (0, (y - kx)^{-1} p, \mathrm{Id})$$

induces the trivialization of $E_{(x,\bar{x},p,q)}$ restricted to $(\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1$. Note that these two trivializations agree at the point (∞,∞) and extend the trivialization of E restricted to two infinite lines. Now we can construct $\eta(E)$. We just have to calculate the transition function $(\tau_1^{-1} \circ \tau_2)_{|1 \times (\mathbb{P}^1 \setminus \{0,\infty\})}$ it is the point in Gr_{GL_d} corresponding to $E_{|1 \times (\mathbb{P}^1 \setminus \{0,\infty\})}$ and the trivialization induced by τ_1. It is easy to see that the corresponding point is

$$\eta(E) = 1 + q(\bar{x} - z)^{-1}(x-1)^{-1} p.$$

This finishes the proof.

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